Sharp Adams-type inequalities in $\mathbb{R}^N$

Bernhard Ruf

Abstract:

Adams' inequality is a generalization of the Trudinger-Moser inequalitiy to higher order Sobolev spaces. In particular, for the second order Sobolev space $W_0^{2,2}(\Omega)$ it states, for bounded domains $\Omega \subset \mathbb{R}^4$, that the supremum of $\int_\Omega e^{32\pi^2 u^2}$ over all functions $u \in W_0^{2,\,2}(\Omega)$ with $\|\Delta u\|_2 \le 1$ is bounded by a constant depending on Ω only. This bound becomes infinite for unbounded domains and in particular for $\mathbb{R}^4$.
We prove that if $\|\Delta u\|_2$ is replaced by a suitable equivalent norm, namely $\|u\| := \| - \Delta u + u\|_2$, then the supremum of $\int_\Omega (e^{32\pi^2 u^2} - 1)$ over all functions $u \in W_0^{2,\,2}(\Omega)$ with $\|u\| \le 1$ is bounded by a constant independent of the domain Ω. Furthermore, we generalize this result to any Sobolev space $W_0^{m,\,\frac{N}{m}}(\Omega)$ with $\Omega \subseteq \mathbb{R}^N$ and m an even integer less than N.