

FROM FORMS TO SEMIGROUPS

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Introduction

Form methods give a very efficient tool to solve evolutionary problems on Hilbert space. They were developed by T. Kato [Kat80] and, in slightly different language by J.L. Lions. In these lectures we give an introduction based on [AtE09], where some generalization is introduced. The new setting is particularly efficient for the Dirichlet-to-Neumann operator and degenerate equations. Besides this we give several other examples. These lectures start by an introduction to holomorphic semigroups. Instead of the contour argument found in the literature, we give a more direct argument based on the Hille-Yosida theorem.

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1 The Hille–Yosida Theorem

A C_0 -semigroup on a Banach space X is a mapping $T : (0, \infty) \rightarrow \mathcal{L}(X)$ satisfying

$$\begin{aligned} T(t+s) &= T(t)T(s) \\ \lim_{t \downarrow 0} T(t)x &= x \quad (x \in X) . \end{aligned}$$

The *generator* A of such a C_0 -semigroup is defined by

$$\begin{aligned} D(A) &:= \{x \in X : \lim_{t \downarrow 0} \frac{T(t)x - x}{t} \text{ exists}\} \\ Ax &:= \lim_{t \downarrow 0} \frac{T(t)x - x}{t} . \end{aligned}$$

Thus the domain $D(A)$ of A is a subspace of X and $A : D(A) \rightarrow X$ is linear. One can show that $D(A)$ is dense in X . The main interest in semigroups lies in the associated Cauchy problem

$$(CP) \begin{cases} \dot{u}(t) &= Au(t) & (t > 0) \\ u(0) &= x . \end{cases}$$

Indeed, if A is the generator of a C_0 -semigroup, then given $x \in X$, the function $u(t) := T(t)x$ is the unique *mild* solution of (CP) ; i.e.

$$u \in C([0, \infty); X) , \quad \int_0^t u(s) \, ds \in D(A)$$

for all $t > 0$ and

$$\begin{aligned} u(t) &= x + A \int_0^t u(s) \, ds \\ u(0) &= x . \end{aligned}$$

If $x \in D(A)$, then u is a *classical solution*; i.e. $u \in C^1([0, \infty); X)$, $u(t) \in D(A)$ for all $t \geq 0$ and $\dot{u}(t) = Au(t)$ for all $t > 0$. Conversely, if for each $x \in X$ there exists a unique mild solution of (CP) , then A generates a C_0 -semigroup [ABHN01, Theorem 3.1.12].

1 The Hille-Yosida Theorem

In view of this characterization of well-posedness, it is of big interest to decide whether a given operator generates a C_0 -semigroup. A positive answer is given by the famous Hille-Yosida Theorem.

Theorem 1.1. (*Hille-Yosida (1948)*). *Let A be an operator on X . The following are equivalent.*

(i) *A generates a contractive C_0 -semigroup;*

(ii) *the domain of A is dense, $\lambda - A$ is invertible for $\lambda > 0$ and $\|\lambda(\lambda - A)^{-1}\| \leq 1$.*

Here we call a semigroup T *contractive* if $\|T(t)\| \leq 1$ for all $t > 0$. By $\lambda - A$ we mean the operator with domain $D(A)$ given by $(\lambda - A)x := \lambda x - Ax$ ($x \in D(A)$). So the condition in (ii) means that $\lambda - A : D(A) \rightarrow X$ is bijective and $\|\lambda(\lambda - A)^{-1}x\| \leq \|x\|$ for all $\lambda > 0, x \in X$. If X is reflexive, then this existence of the *resolvent* $(\lambda - A)^{-1}$ and the contractivity $\|\lambda(\lambda - A)^{-1}\| \leq 1$ imply already that the domain is dense [ABHN01, Theorem 3.3.8].

Yosida's proof is based on the Yosida-approximation: Assuming (ii), one easily sees that

$$\lim_{\lambda \rightarrow \infty} \lambda(\lambda - A)^{-1}x = x \quad (x \in D(A)) ,$$

i.e. $\lambda(\lambda - A)^{-1}$ converges strongly to the identity as $\lambda \rightarrow \infty$. This implies that

$$A_\lambda := A\lambda(\lambda - A)^{-1} = \lambda^2(\lambda - A)^{-1} - \lambda$$

approximates A as $\lambda \rightarrow \infty$ in the sense that

$$\lim_{\lambda \rightarrow \infty} A_\lambda x = Ax \quad (x \in D(A)) .$$

The operator A_λ is bounded, so one may define

$$e^{tA_\lambda} := \sum_{n=0}^{\infty} \frac{t^n A_\lambda^n}{n!}$$

by the power series. Note that $\|\lambda^2(\lambda - A)^{-1}\| \leq \lambda$. Since

$$e^{tA_\lambda} = e^{-\lambda t} e^{t\|\lambda^2(\lambda - A)^{-1}\|} ,$$

it follows that

$$\|e^{tA_\lambda}\| \leq e^{-\lambda t} e^{t\|\lambda^2(\lambda - A)^{-1}\|} \leq 1 .$$

Yosida's proof consists in showing that for $x \in X$ the family $(e^{tA_\lambda}x)_{\lambda>0}$ is a Cauchy net as $\lambda \rightarrow \infty$. Then the C_0 -semigroup generated by A is given by

$$T(t)x := \lim_{\lambda \rightarrow \infty} e^{tA_\lambda}x \quad (t > 0)$$

for all $x \in X$. We will come back to this formula when we talk about holomorphic semigroups.

Remark 1.2. Hille's independent proof is based on Euler's formula for the exponential function. Note that putting $t = \frac{1}{\lambda}$ one has

$$\lambda(\lambda - A)^{-1} = (I - tA)^{-1}.$$

Hille showed that

$$T(t)x := \lim_{n \rightarrow \infty} (I - \frac{t}{n}A)^{-n}x$$

exists for all $x \in X$, see [Kat80].

1 *The Hille-Yosida Theorem*

2 Holomorphic semigroups

A C_0 -semigroup is defined on the real half-line $(0, \infty)$ with values in $\mathcal{L}(X)$. It is useful to study when extensions to a sector

$$\Sigma_\theta := \{re^{i\alpha} : r > 0, |\alpha| < \theta\}$$

for some $\theta \in (0, \pi/2]$ exist. In this section X is a complex Banach space.

Definition 2.1. A C_0 -semigroup T is called *holomorphic* if there exists $\theta \in (0, \pi/2]$ and a holomorphic extension

$$\tilde{T} : \Sigma_\theta \rightarrow \mathcal{L}(X)$$

of T which is locally bounded; i.e.

$$\sup_{\substack{z \in \Sigma_\theta \\ |z| \leq 1}} \|\tilde{T}(z)\| < \infty .$$

If $\|\tilde{T}(z)\| \leq 1 (z \in \Sigma_\theta)$, then we call T a *holomorphic sectorially contractive* C_0 -semigroup (of angle θ , if we want to make precise the angle).

The holomorphic extension \tilde{T} automatically has the semigroup property

$$\tilde{T}(z_1 + z_2) = \tilde{T}(z_1)\tilde{T}(z_2) \quad (z_1, z_2 \in \Sigma_\theta) .$$

Because of the boundedness assumption it follows that

$$\lim_{\substack{z \rightarrow 0 \\ z \in \Sigma_\theta}} \tilde{T}(z)x = x \quad (x \in X) .$$

These properties are easy to see. In the sequel we will omit the \sim and denote the extension \tilde{T} simply by T . We should add a remark on vector-valued holomorphic functions.

Remark 2.2. If Y is a Banach space, $\Omega \subset \mathbb{C}$ open, then a function $f : \Omega \rightarrow Y$ is called *holomorphic* if

$$f'(z) = \lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h}$$

2 Holomorphic semigroups

exists in the norm of Y for all $z \in \Omega$ and $f' : \Omega \rightarrow Y$ is continuous. It follows as in the scalar case that f is analytic. It is remarkable that holomorphy is the same as weak holomorphy (first observed by Grothendieck): A function $f : \Omega \rightarrow Y$ is holomorphic if and only if

$$y' \circ f : \Omega \rightarrow \mathbb{C}$$

is holomorphic for all $y' \in Y'$. In our context the space Y is $\mathcal{L}(X)$, the space of all bounded linear operators on X with the operator norm. If the function f is bounded it suffices to test holomorphy with few functionals. We say that a subspace $W \subset Y'$ *seperates points* if for $x \in Y$,

$$\langle y', x \rangle = 0 \text{ for } y' \in W \text{ implies } x = 0 .$$

Assume that $f : \Omega \rightarrow Y$ is bounded such that $y' \circ f$ is holomorphic for all $y' \in W$ where W is a separating subspace of Y' . Then f is holomorphic. This result is due to [AN00], see also [ABHN01, Theorem A7]. In particular, if $Y = \mathcal{L}(X)$, then a bounded function $f : \Omega \rightarrow \mathcal{L}(X)$ is holomorphic if and only if $\langle x', f(\cdot)x \rangle$ is holomorphic for all x in a dense subspace of X and all x' in a separating subspace of X' .

We recall a special form of Vitali's Theorem (see [AN00], [ABHN01, Theorem A5]). Let $\Omega \subset \mathbb{C}$ be connected.

Theorem 2.3. (*Vitali*). *Let $f_n : \Omega \rightarrow \mathcal{L}(X)$ be holomorphic such that*

$$a) \|f_n(z)\| \leq M \quad (z \in \Omega, n \in \mathbb{N}) \text{ and}$$

$$b) f(z)x := \lim_{n \rightarrow \infty} f_n(z)x$$

exists for all $x \in X$ and all $z \in \Omega_0$, where $\Omega_0 \subset \Omega$ contains a stationary non-sequence with limit point in Ω , i.e. there exist $z_k \in \Omega_0, z_0 \in \Omega, z_k \neq z_0$ ($k \in \mathbb{N}$), such that $\lim_{k \rightarrow \infty} z_k = z_0$. Then

$$f(z)x := \lim_{n \rightarrow \infty} f_n(z)x$$

exists for all $x \in X, z \in \Omega$ and $f : \Omega \rightarrow \mathcal{L}(X)$ is holomorphic.

Now we want to give a simple characterization of holomorphic sectorially contractive semigroups. Assume that A is a densely defined operator on X such that $(\lambda - A)^{-1}$ exists and

$$\|(\lambda - A)^{-1}\| \leq 1 \quad (\lambda \in \Sigma_\theta) ,$$

where $0 < \theta \leq \pi/2$. Let $z \in \Sigma_\theta$. Then for $\lambda > 0$,

$$(zA)_\lambda = zA_{\frac{\lambda}{z}}$$

is holomorphic in z . For each $z \in \Sigma_\theta$, the operator zA is m -dissipative. By the Hille-Yosida Theorem

$$T(z)x := \lim_{\lambda \rightarrow \infty} e^{(zA)\lambda} x$$

exists for all $x \in X, z \in \Sigma_\theta$. Since $e^{(zA)\lambda} = e^{zA\lambda/z}$ is holomorphic, $T : \Sigma_\theta \rightarrow \mathcal{L}(X)$ is holomorphic by Vitali's Theorem. If $t > 0$, then

$$T(t) = \lim_{\lambda \rightarrow \infty} e^{tA\lambda/t} = T_A(t)$$

where T_A is the semigroup generated by A . Since $T_A(t+s) = T_A(t)T_A(s)$, it follows from analytic continuation that

$$T(z_1 + z_2) = T(z_1)T(z_2) \quad (z_1, z_2 \in \Sigma_\theta) .$$

Thus A generates a sectorially contractive holomorphic C_0 -semigroup of angle θ on X . One sees as above that

$$T_{zA}(t) = T(z t)$$

for all $t > 0, z \in \Sigma_\theta$. We have shown the following.

Theorem 2.4. *Let A be a densely defined operator on $X, \theta \in (0, \pi/2]$. The following are equivalent.*

- (i) *A generates a sectorially contractive holomorphic C_0 -semigroup of angle θ ;*
- (ii) *$(\lambda - A)^{-1}$ exists for $\lambda \in \Sigma_\theta$ and*

$$\|\lambda(\lambda - A)^{-1}\| \leq 1 \quad (\lambda \in \Sigma_\theta) .$$

3 The Lumer–Phillips Theorem

Let H be a Hilbert space over $\mathbb{K} = \mathbb{R}$ or \mathbb{C} . An operator A on H is called *accretive* or *monotone* if

$$\operatorname{Re}(Ax|x) \geq 0 \quad (x \in D(A)) .$$

Theorem 3.1. (*Lumer-Phillips*). *Let A be an operator on H . The following are equivalent.*

- (i) $-A$ generates a contraction semigroup;
- (ii) A is accretive and $(I + A)$ is surjective.

Accretivity of A can be reformulated by the condition

$$\|(\lambda + A)x\| \geq \|\lambda x\| \quad (\lambda > 0, x \in D(A)) .$$

Thus if $\lambda + A$ is surjective, then $(\lambda + A)$ is invertible and $\|\lambda(\lambda + A)^{-1}\| \leq 1$. We also say that A is *m-accretive* if condition (ii) is satisfied. If A is *m-accretive* and $\mathbb{K} = \mathbb{C}$, then one can easily see that $\lambda + A$ is invertible for all $\lambda \in \mathbb{C}$ satisfying $\operatorname{Re} \lambda > 0$ and

$$\|(\lambda + A)^{-1}\| \leq \frac{1}{\operatorname{Re} \lambda} .$$

Due to the reflexivity of Hilbert spaces, each *m-accretive* operator A is densely defined (see [ABHN01, Proposition 3.3.8]). Now we want to reformulate the Lumer-Phillips Theorem for generators of semigroups which are contractive on a sector.

Theorem 3.2. (*generators of sectorially contractive semigroups*). *Let A be an operator on a complex Hilbert space H and let $\theta \in (0, \frac{\pi}{2})$. The following are equivalent.*

- (i) $-A$ generates a holomorphic C_0 -semigroup which is contractive on the sector Σ_θ .
- (ii) $e^{\pm i\theta} A$ is accretive and $I + A$ is surjective.

3 The Lumer-Phillips Theorem

Proof. (ii) \Rightarrow (i). Since $e^{\pm i\theta}A$ is accretive zA is accretive for all $z \in \Sigma_\theta$. Since $(I + A)$ is surjective, the operator A is m -accretive. Thus $(\lambda + A)$ is invertible whenever $\operatorname{Re} \lambda > 0$. Consequently $(I + z) = z(z^{-1} + A)$ is invertible for all $z \in \Sigma_\theta$. Thus zA is m -accretive for all $z \in \Sigma_\theta$. Now (i) follows from Theorem ??.

(i) \Rightarrow (ii). If $-A$ generates a holomorphic semigroup which is contractive on Σ_θ , then $e^{i\alpha}A$ generates a contraction semigroup for $|\alpha| \leq \theta$. Hence $e^{i\alpha}A$ is m -accretive for $|\alpha| \leq \theta$. \square

4 Forms: the complete case

We recall one of our most efficient tool to solve equations, the Lax-Milgram lemma which is just a non-symmetric generalization of the Riesz-Fréchet representation Theorem from 1905.

Lemma 4.1. (*Lax-Milgram 1954*). *Let V be a Hilbert space over $\mathbb{K} = \mathbb{R}$ or \mathbb{C} and let $a : V \times V \rightarrow \mathbb{K}$ be sesquilinear, continuous and coercive, i.e.*

$$\operatorname{Re} a(u) \geq \alpha \|u\|_V^2 \quad (u \in V)$$

for some $\alpha > 0$. Let $\varphi \in V'$ be a continuous anti-linear form, i.e. $\varphi : V \rightarrow \mathbb{K}$ is continuous and satisfies $\varphi(u + v) = \varphi(u) + \varphi(v)$, $\varphi(\lambda u) = \bar{\lambda}\varphi(u)$ ($u, v \in V, \lambda \in \mathbb{K}$). Then there is a unique $u \in V$ such that

$$a(u, v) = \varphi(v) \quad (v \in V) .$$

Of course, to say that a is continuous means that

$$|a(u, v)| \leq M \|u\|_V \|v\|_V \quad (u, v \in V)$$

for some constant M . We let $a(u) := a(u, u)$ for $u \in V$.

In general, the range condition in the Hille-Yosida Theorem is difficult to prove. However, if we look at operators associated with a form, the Lax-Milgram Lemma implies automatically the range condition. We describe now our general setting in the complete case. Given is a Hilbert space V over $\mathbb{K} = \mathbb{R}$ or \mathbb{C} and a continuous, coercive sesquilinear form

$$a : V \times V \rightarrow \mathbb{K} .$$

Moreover, we assume that H is another Hilbert space over \mathbb{K} and $j : V \rightarrow H$ is a continuous linear mapping with dense image. Now we associate an operator A on H with the pair (a, j) in the following way: Given $x, y \in H$ we say that $x \in \mathcal{D}(A)$ and $Ax = y$ if there exists $u \in V$ such that $j(u) = x$ and

$$a(u, v) = (y|j(v))_H \text{ for all } v \in V .$$

4 Forms: the complete case

We first show that A is univocal: Assume that there exist $u_1, u_2 \in V$, $y_1, y_2 \in H$ such that

$$\begin{aligned} j(u_1) &= j(u_2) = x && \text{and} \\ a(u_1, v) &= (y_1|j(v))_H && \text{for all } v \in V \\ \text{and } a(u_2, v) &= (y_2|j(v))_H && \text{for all } v \in V . \end{aligned}$$

Then $a(u_1 - u_2, v) = (y_1 - y_2|j(v))_H$ for all $v \in V$. Since $j(u_1 - u_2) = 0$, taking $v := u_1 - u_2$ gives $a(u_1 - u_2, u_1 - u_2) = 0$. Since a is coercive, it follows that $u_1 = u_2$. It follows that $(y_1|j(v))_H = (y_2|j(v))_H$ for all $v \in V$. Since j has dense image, it follows that $y_1 = y_2$. \square

It is clear from the definition that $A : D(A) \rightarrow H$ is linear. Our main result is the following generation theorem. We first assume that $\mathbb{K} = \mathbb{C}$.

Theorem 4.2. (*generation theorem in the complete case*). *The operator $-A$ generates a sectorially contractive holomorphic C_0 -semigroup T . If a is symmetric, then A is selfadjoint.*

Proof. Letting $M \geq 0$ be the constant of continuity and $\alpha > 0$ the constant of coerciveness as before, we have

$$\frac{|\operatorname{Im} a(v)|}{\operatorname{Re} a(v)} \leq \frac{M\|v\|_V^2}{\alpha\|v\|_V^2} = \frac{M}{\alpha}$$

for all $v \in V$. Thus there exists $\theta' \in [0, \frac{\pi}{2})$ such that

$$a(v) \in \Sigma_{\theta'} \text{ for all } v \in V .$$

Let $x \in D(A)$. There exists $u \in V$ such that $x = j(u)$ and $a(u, v) = (Ax|j(v))_H$ for all $v \in V$. In particular, $(Ax|x)_H = a(u) \in \Sigma_{\theta'}$. It follows that $e^{\pm i\theta} A$ is accretive where $\theta = \frac{\pi}{2} - \theta'$. In order to prove the range condition, let $y \in H$. Consider the form $b : V \times V \rightarrow \mathbb{C}$ given by

$$b(u, v) = a(u, v) + (j(u)|j(v))_H .$$

Then b is continuous and coercive. Let $y \in H$. Then $\varphi(v) := (y|j(v))_H$ defines a continuous antilinear form φ on V . By the Lax-Milgram lemma 4.1 there exists a unique $u \in V$ such that

$$b(u, v) = \varphi(v) \quad (v \in V) .$$

Hence $(y|j(v))_H = a(u, v) + (j(u)|j(v))_H$; i.e. $a(u, v) = (y - j(u)|j(v))_H$ ($v \in V$). This means that $x := j(u) \in D(A)$ and $Ax = y - x$. \square

The result is also valid in real Banach spaces. If T is a C_0 -semigroup on a real Banach space X , then the \mathbb{C} -linear extension $T_{\mathbb{C}}$ of T on the complexification $X_{\mathbb{C}} := X \oplus iX$ of X is a C_0 -semigroup given by $T_{\mathbb{C}}(t)(x + iy) := T(t)x + iT(t)y$. We call T *holomorphic* if $T_{\mathbb{C}}$ is holomorphic. The generation theorem above remains true on real Banach spaces. In order to formulate a final result we want also allow a rescaling. Let X be a Banach space over \mathbb{K} and T be a C_0 -semigroup on X with generator A . Then for $\omega \in \mathbb{K}$

$$T_{\omega}(t) := e^{\omega t}T(t)$$

defines a C_0 -semigroup whose generator is $A + \omega$. Using this we obtain now the following general generation theorem in the complete case. Let V, H be Hilbert spaces over \mathbb{K} and $j : V \rightarrow H$ linear with dense image. Let $a : V \times V \rightarrow \mathbb{K}$ be sesquilinear and continuous. We call the form a *j -elliptic* if there exists $\omega \in \mathbb{R}$ such that

$$(4.1) \quad \operatorname{Re} a(u) + \omega \|j(u)\|_H^2 \geq \alpha \|u\|_V^2$$

for all $u \in V$. Then we define the operator A associated with (j, a) as before by

$$\begin{aligned} D(A) &:= \{x \in H : \exists u \in V, y \in H \text{ such that} \\ &\quad j(u) = x \text{ and } a(u, v) = (y|j(v))_H \text{ for all } v \in H\} \\ Ax &:= y \text{ where } j(u) = x, a(u, v) = (y|j(v))_H \quad (v \in V) . \end{aligned}$$

Theorem 4.3. *The operator defined in this way is univocal. Moreover, $-A$ generates a holomorphic C_0 -semigroup on H .*

Remark 4.4. The form a satisfies condition (4.1) if and only if the form a_{ω} given by

$$a_{\omega}(u, v) = a(u, v) + \omega (j(u)|j(v))_H$$

is coercive. If T_{ω} denotes the semigroup associated with a_{ω} and T the semigroup associated with a , then

$$T_{\omega}(t) = e^{-\omega t}T(t) \quad (t \geq 0)$$

as is easy to see.

4 *Forms: the complete case*

5 The Stokes Operator

In this section we show as an example that the Stokes operator is selfadjoint and generates a holomorphic C_0 -semigroup. Let $\Omega \subset \mathbb{R}^d$ be a bounded open set. We first discuss the Dirichlet-Laplacian.

Theorem 5.1. (*Dirichlet Laplacian*). *Let $H = L^2(\Omega)$ and define the operator Δ^D on $L^2(\Omega)$ by*

$$\begin{aligned} D(\Delta^D) &= \{u \in H_0^1(\Omega) : \Delta u \in L^2(\Omega)\} \\ \Delta^D u &:= \Delta u . \end{aligned}$$

Then Δ^D is selfadjoint and generates a holomorphic C_0 -semigroup on $L^2(\Omega)$.

Proof. Define $a : H_0^1(\Omega) \times H_0^1(\Omega) \rightarrow \mathbb{R}$ by $a(u, v) = \int_{\Omega} \nabla u \nabla v$. Then a is clearly continuous. Poincaré's inequality says that a is coercive. Consider the injection j of $H_0^1(\Omega)$ into $L^2(\Omega)$. Let A be the operator associated with (a, j) . We show that $A = -\Delta^D$. In fact, let $u \in D(A)$, $Au = f$. Then $\int_{\Omega} \nabla u \nabla v = \int_{\Omega} f v$ for all $v \in H_0^1(\Omega)$. Taking in particular $v \in C_c^\infty(\Omega)$ we see that $-\Delta u = f$. Conversely, let $u \in H_0^1(\Omega)$ such that $f := -\Delta u \in L^2(\Omega)$. Then $\int_{\Omega} f \varphi = \int_{\Omega} \nabla u \nabla \varphi = a(u, \varphi)$ for all $\varphi \in C_c^\infty(\Omega)$. This is just the definition of the weak partial derivatives in $H^1(\Omega)$. Since $C_c^\infty(\Omega)$ is dense in $H_0^1(\Omega)$, it follows that $\int_{\Omega} f v = a(u, v)$ for all $v \in H_0^1(\Omega)$. Thus $u \in D(A)$ and $Au = f$. \square

For our treatment of the Stokes operator it will be useful to consider the Dirichlet Laplacian also in $L^2(\Omega)^d = L^2(\Omega) \oplus \dots \oplus L^2(\Omega)$.

Theorem 5.2. *Define the symmetric form*

$$\begin{aligned} a : H_0^1(\Omega)^d \times H_0^1(\Omega)^d &\rightarrow \mathbb{R} \text{ by} \\ a(u, v) = \int_{\Omega} \nabla u \nabla v &:= \sum_{j=1}^d \int_{\Omega} \nabla u_j \nabla v_j \end{aligned}$$

5 The Stokes Operator

where $u = (u_1, \dots, u_d)$. Then a is continuous and coercive. Let $j : H_0^1(\Omega)^d \rightarrow L^2(\Omega)^d$ be the identity. The operator A associated with (a, j) on $L^2(\Omega)^d$ is given by

$$\begin{aligned} D(A) &= \{u \in H_0^1(\Omega)^d : \Delta u_j \in L^2(\Omega), j = 1, \dots, d\} \\ Au &= (-\Delta u_1, \dots, -\Delta u_d) =: -\Delta u . \end{aligned}$$

We call $\Delta^D := -A$ the Dirichlet Laplacian on $L^2(\Omega)^d$.

In order to define the Stokes operator we need some preparation. Let $\mathcal{D}(\Omega) := C_c^\infty(\Omega)^d$ and let $\mathcal{D}_0(\Omega) := \{\varphi \in \mathcal{D}(\Omega) : \operatorname{div} \varphi = 0\}$, where $\operatorname{div} \varphi = \partial_1 \varphi_1 + \dots + \partial_d \varphi_d$, $\varphi = (\varphi_1, \dots, \varphi_d)$. By $\mathcal{D}(\Omega)'$ we denote the dual space of $\mathcal{D}(\Omega)$ (with the usual topology). Each element S of $\mathcal{D}(\Omega)'$ can be written in a unique way as $S = (S_1, \dots, S_d)$ with $S_j \in C_c^\infty(\Omega)'$ so that

$$\langle S, \varphi \rangle = \sum_{j=1}^d \langle S_j, \varphi_j \rangle$$

for $\varphi = (\varphi_1, \dots, \varphi_d) \in \mathcal{D}(\Omega)$.

We say that $S \in H^{-1}(\Omega)$ if there exists a constant such that

$$|\langle S, \varphi \rangle| \leq c \left(\int |\nabla \varphi|^2 \right)^{\frac{1}{2}} \quad (\varphi \in \mathcal{D}(\Omega))$$

where $|\nabla \varphi|^2 = (|\nabla \varphi_1|^2 + \dots + |\nabla \varphi_d|^2)$. For the remainder of this section we assume that Ω has Lipschitz boundary. We need the following result (see [Tem84, Remark 1.9, p. 14]).

Theorem 5.3. *Let $T \in H^{-1}(\Omega)$. The following are equivalent.*

- (i) $\langle T, \varphi \rangle = 0$ for all $\varphi \in \mathcal{D}_0$;
- (ii) there exists $p \in L^2(\Omega)$ such that $T = \nabla p$.

Note that condition (ii) means that

$$\begin{aligned} \langle T, \varphi \rangle &= \sum_{j=1}^d \langle \partial_j p, \varphi_j \rangle \\ &= - \sum_{j=1}^d \langle p, \partial_j \varphi_j \rangle \\ &= \langle p, \operatorname{div} \varphi \rangle . \end{aligned}$$

From this, the implication (ii) \Rightarrow (i) is obvious. We omit the other implication.

Now we consider the real Hilbert space $L^2(\Omega)^d$ with scalar product

$$(f|g) = \sum_{j=1}^d (f_j|g_j)_{L^2(\Omega)} = \sum_{j=1}^d \int_{\Omega} f_j g_j .$$

We denote by

$$H := \mathcal{D}_0^{\perp\perp} = \overline{\mathcal{D}_0}$$

the closure of \mathcal{D}_0 in $L^2(\Omega)^d$. We call H the space of all *divergence free vectors* in $L^2(\Omega)^d$. The orthogonal projection P from $L^2(\Omega)^d$ onto H is called the *Helmholtz projection*. Now let V be the closure of \mathcal{D}_0 in $H^1(\Omega)^d$. Thus $V \subset H_0^1(\Omega)^d$ and $\operatorname{div} u = 0$ for all $u \in V$. One can actually show that

$$V = \{u \in H_0^1(\Omega)^d : \operatorname{div} u = 0\} .$$

We define the form

$$\begin{aligned} a : V \times V &\rightarrow \mathbb{R} \\ a(u, v) &= \sum_{j=1}^d (\nabla u_j | \nabla v_j)_{L^2(\Omega)} \\ u &= (u_1, \dots, u_d), v = (v_1, \dots, v_d) \in V . \end{aligned}$$

Then a is continuous and coercive. The space V is dense in H since it contains \mathcal{D}_0 . We consider the identity $j : V \rightarrow H$. Let A be the operator associated with (a, j) . Then A is selfadjoint and $-A$ generates a holomorphic C_0 -semigroup. The operator can be described as follows.

Theorem 5.4. *The operator A has the domain*

$$D(A) = \{u \in V : \exists \pi \in L^2(\Omega) \text{ such that } -\Delta u + \nabla \pi \in H\}$$

and is given by

$$Au = -\Delta u + \nabla \pi .$$

If $u \in H_0^1(\Omega)^d$, then $\Delta u \in H^{-1}(\Omega)$. In fact, for $\varphi \in \mathcal{D}(\Omega)$,

$$| \langle -\Delta u, \varphi \rangle | = | - \langle u, \Delta \varphi \rangle | = \left| \sum_{j=1}^d \int_{\Omega} \nabla u_j \nabla \varphi_j \right| \leq \|u\|_{H_0^1(\Omega)^d} \|\varphi\|_{H_0^1(\Omega)^d} .$$

Proof of Theorem 5.4. Let $u \in D(A)$, $Au = f$. Then $f \in H$, $u \in V$ and $a(u, v) = (f|v)_H$ for all $v \in V$. Thus, the distribution $\Delta u \in H^{-1}(\Omega)$ coincides with f on \mathcal{D}_0 .

5 The Stokes Operator

By Theorem 5.3 there exists $\pi \in L^2(\Omega)'$ such that $-\Delta u + \nabla \pi = f$. Conversely, let $u \in V, f \in H, \pi \in L^2(\Omega)$ such that $-\Delta u + \nabla \pi = f$ in $\mathcal{D}(\Omega)'$. Then for $\varphi \in \mathcal{D}_0$

$$a(u, \varphi) = \int_{\Omega} \nabla u \nabla \varphi = \int_{\Omega} \nabla u \nabla \varphi + \langle \nabla \pi, \varphi \rangle = (f|\varphi)_{L^2(\Omega)^d} .$$

Since \mathcal{D}_0 is dense in V , it follows that $a(u, \varphi) = (f|\varphi)_{L^2(\Omega)^d}$ for all $\varphi \in V$. Thus, $u \in D(A)$ and $Au = f$. \square

The operator A is called the *Stokes operator*. We refer to [Mon06] for this approach and further results on the Navier-Stokes equation. We conclude this section by giving an example where j is not injective. Further examples will be seen in the sequel.

Proposition 5.5. *Let \tilde{H} be a Hilbert space and $H \subset \tilde{H}$ a closed subspace. Denote by P the orthogonal projection onto H . Let \tilde{V} be a Hilbert space which is continuously and densely embedded into \tilde{H} and let $a : \tilde{V} \times \tilde{V} \rightarrow \mathbb{R}$ be a continuous, coercive form. Denote by A the operator on \tilde{H} associated with (a, j) where j is the injection of \tilde{V} into \tilde{H} and let B be the operator on H associated with $(a, P \circ j)$. Then*

$$\begin{aligned} D(B) &= \{Pw : w \in D(A), Aw \in H\} , \\ BPw &= Aw . \end{aligned}$$

This is easy to see. In the context considered in this section we obtain the following example.

Example 5.6. Let $\tilde{H} = L^2(\Omega)^d, H = \overline{\mathcal{D}_0}, \tilde{V} := H_0^1(\Omega)^d$ and $a(u, v) = \int_{\Omega} \nabla u \nabla v, j : \tilde{V} \rightarrow \tilde{H}, j(u) = u$. Then the operator associated with (a, j) is $A = -\Delta_D$ as we have seen in Theorem 5.2. Now let P be the Helmholtz projection and B the operator associated with (a, P) . Then

$$\begin{cases} D(B) &= \{u \in H : \exists \pi \in L^2(\Omega), u + \nabla \pi \in D(\Delta^D), \Delta(u + \nabla \pi) \in H\} \\ Bu &= \Delta(u + \nabla \pi) . \end{cases}$$

This follows directly from Proposition 5.5. Thus, the operator B is selfadjoint and generates a holomorphic semigroup.

6 From forms to semigroups: the incomplete case

In the preceding sections we considered forms which were defined on a Hilbert space V . Now we want to study a purely algebraic condition. At first we consider the complex case. Let H be a complex Hilbert space. A *sectorial form* on H is a sesquilinear form

$$a : D(a) \times D(a) \rightarrow \mathbb{C}$$

together with a linear mapping $j : D(a) \rightarrow H$ with dense image such that there exist $\omega \geq 0, \theta \in [0, \pi/2)$ such that

$$a(u) + \omega(\|j(u)\|_H^2) \in \Sigma_\theta \quad (u \in D(a)) .$$

If $\omega = 0$, then we call the form *0-sectorial*. To such a form, we associate an operator A on H by stipulating for $x, y \in H : x \in D(A), Ax = y :\Leftrightarrow$ there exist $u_n \in D(a)$ such that

- a) $\lim_{n \rightarrow \infty} j(u_n) = x$ in H ;
- b) $\sup_{n \in \mathbb{N}} \operatorname{Re} a(u_n) < \infty$;
- c) $\lim_{n \rightarrow \infty} a(u_n, v) = (y|j(v))_H$ for all $v \in D(a)$.

Theorem 6.1. *The operator A associated with a sectorial form (a, j) is univocal and $-A$ generates a holomorphic C_0 -semigroup on H .*

The proof of the theorem consists in a reduction to the complete case by considering an appropriate completion of $D(a)$. Here it is important that in Theorem 4.2 a non-injective mapping j is allowed. We refer to [AtE09]. If $C \subset H$ is a closed convex set, we say that C is *invariant* under the semigroup T if

$$T(t)C \subset C \quad (t \geq 0) .$$

Invariant sets are important to study positivity, L^∞ -contractivity, and many more properties. If the semigroup is associated with a form, then the following criterion is convenient.

Theorem 6.2. (*invariance*). *Let $C \subset H$ be a closed convex set and let P be the orthogonal projection onto C . The semigroup T associated with a sectorial form (a, j) on H leaves C invariant if and only if for each $u \in D(a)$ there exists a sequence $(w_n)_{n \in \mathbb{N}}$ in $D(a)$ such that*

- a) $\lim_{n \rightarrow \infty} j(w_n) = Pj(u)$ in H ;
- b) $\limsup_{n \rightarrow \infty} \operatorname{Re} a(w_n, u - w_n) \geq 0$ and
- c) $\sup_{n \in \mathbb{N}} \operatorname{Re} a(w_n) < \infty$.

Corollary 6.3. *Assume that for each $u \in D(a)$, there exists $w \in D(a)$ such that*

$$\begin{aligned} j(w) &= Pj(u) \text{ and} \\ \operatorname{Re} a(w, u - w) &\geq 0 . \end{aligned}$$

Then $T(t)C \subset C$ for all $t \geq 0$.

In this section we want to use the invariance criterion to prove a generation theorem in the incomplete case which is valid in real Hilbert spaces. Let H be a real Hilbert space. A *sectorial* form on H is a bilinear mapping

$$a : D(a) \times D(a) \rightarrow \mathbb{R} ,$$

where $D(a)$ is a real vector space, together with a linear mapping $j : D(a) \rightarrow H$ with dense image satisfying

$$|a(u, v) - a(v, u)| \leq \alpha(a(u) + a(v)) + \omega(\|j(u)\|_H^2 + \|j(v)\|_H^2)$$

for all $u, v \in D(a)$ and some constants $\alpha \geq 0$, $\omega \geq 0$. To such a form we associate an operator A on H by stipulating for $x, y \in H : x \in D(A), Ax = y \Leftrightarrow$ there exist $u_n \in D(a)$ satisfying

- a) $\lim_{n \rightarrow \infty} j(u_n) = x$ in H ;
- b) $\sup_{n \in \mathbb{N}} a(u_n) < \infty$ and
- c) $\lim_{n \rightarrow \infty} a(u_n, v) = (y|j(v))_H$ for all $v \in D(a)$.

Then the following holds:

Theorem 6.4. *The operator A is univocal and $-A$ generates a holomorphic C_0 -semigroup on H .*

Proof. Consider the complexifications $H_{\mathbb{C}} = H \oplus iH$ and $D(a_{\mathbb{C}}) := D(a) + iD(a)$. Letting

$$a_{\mathbb{C}}(u, v) := a(\operatorname{Re} u, \operatorname{Re} v) + a(\operatorname{Im} u, \operatorname{Im} v) + i(a(\operatorname{Re} u, \operatorname{Im} v) + a(\operatorname{Im} u, \operatorname{Re} v))$$

for $u = \operatorname{Re} u + i \operatorname{Im} u, v = \operatorname{Re} v + i \operatorname{Im} v \in D(a_{\mathbb{C}})$, defines a sesquilinear form $a_{\mathbb{C}}$ on $D(a_{\mathbb{C}})$.

Let $J : D(a_{\mathbb{C}}) \rightarrow H_{\mathbb{C}}$ be the \mathbb{C} -linear extension of j . Let

$$b(u, v) = a_{\mathbb{C}}(u, v) + \omega(J(u)|(J(v))_{H_{\mathbb{C}}}) \quad (u, v \in D(a_{\mathbb{C}})) .$$

Then

$$\operatorname{Im} b(u) = a(\operatorname{Im} u, \operatorname{Re} u) - a(\operatorname{Re} u, \operatorname{Im} u)$$

$$\operatorname{Re} b(u) = a(\operatorname{Re} u) + a(\operatorname{Im} u) + \omega(\|j(\operatorname{Re} u)\|_H^2 + \|j(\operatorname{Im} u)\|_H^2) .$$

The assumption implies that $|\operatorname{Im} b(u)| \leq c \operatorname{Re} b(u)$ for some $c > 0$. Consequently, $b(u) \in \Sigma_{\theta}$ for $\theta = \arctan c$. Thus the operator B associated with b generates a C_0 -semigroup $S_{\mathbb{C}}$ on $H_{\mathbb{C}}$. It follows from Corollary 6.3 that H is invariant. The part A_{ω} of B in H is the generator of $S(t) := S_{\mathbb{C}}(t)|_H$. It is easy to see that $A_{\omega} - \omega = A$. \square

Remark 6.5. It is remarkable, and important for some applications, that (b) may be replaced by

$$(b') . \quad \lim_{n, m \rightarrow \infty} a(u_n - u_m) = 0$$

For later purposes we carry over the invariance criterion Theorem 5.3 to the real case.

Corollary 6.6. *Let H be a real Hilbert space and (a, j) a sectorial form on H with associated semigroup T . Let $C \subset H$ be a closed convex set and P the orthogonal projection onto C . Assume that for each $u \in D(a)$ there exists $w \in D(a)$ such that*

$$j(w) = Pj(u) \text{ and } a(w, u - w) \geq 0 .$$

Then $T(t)C \subset C$ for all $t \geq 0$.

We want to formulate a special case of invariance. An operator S on a space $L^p(\Omega)$ is called

$$\begin{aligned} & \text{positive if } [f \geq 0 \quad \text{implies } Sf \geq 0] \text{ and} \\ & \text{submarkovian if } [f \leq 1 \text{ a.e.} \quad \text{implies } Sf \leq 1 \text{ a.e.}] . \end{aligned}$$

Thus, an operator S is submarkovian if and only if it is positive and $\|Sf\|_{\infty} \leq \|f\|_{\infty}$ for all $f \in L^2 \cap L^{\infty}$.

Proposition 6.7. *Consider the real space $H = L^2(\Omega)$ and a sectorial form a on H . Assume that for each $u \in D(a)$ one has $u \wedge 1 \in D(a)$ and*

$$(u \wedge 1, (u - 1)^+) \geq 0 .$$

Then the semigroup T associated with a is submarkovian.

Proof. The set $C := \{u \in L^2(\Omega) : u \leq 1 \text{ a.e.}\}$ is closed and convex. The orthogonal projection P onto C is given by $Pu = u \wedge 1$. Thus $u - Pu = (u - 1)^+$ and the result follows from Corollary 6.3. \square

We conclude this section by some references to the literature. In many text books, for example [Kat80],[Ouh05] one finds the notion of a sectorial form a on a complex Hilbert space H . By this one understands a sesquilinear form $a : D(a) \times D(a) \rightarrow \mathbb{C}$ where $D(a)$ is a dense subspace of H such that $a(u) + \omega \|u\|_H^2 \in \Sigma_\theta$ for all $u \in D(a)$ and some $\theta \in [0, \pi/2)$ and some $\omega \geq 0$. Then

$$\|u\|_a := (\operatorname{Re} a(u) + (\omega + 1)\|u\|_H^2)^{1/2}$$

defines a norm on $D(a)$. The form is called *closed* if $D(a)$ is complete for this norm. This corresponds to our complete case with $V = D(a)$ and j the identity. If the form is not closed, then one may consider the completion V of $D(a)$. Since the injection $D(a) \rightarrow H$ is continuous for the norm $\|\cdot\|_a$, it has a continuous extension $j : V \rightarrow H$. This extension may be injective or not. The form is called *closable* if j is injective. In the literature only for closable forms generation theorems are given, see [AtE09] for precise references. The results above show that the notion of closability is not needed. There is a unique correspondance between sectorially quasi contractive holomorphic semigroups and closed sectorial forms (see [AtE09] and [Kat80]). One loses uniqueness if one considers forms which are merely closable or our general setting allowing arbitrary maps $j : D(a) \rightarrow H$ with dense image. However, examples show that in many cases a natural operator is obtained by this general framework.

7 Degenerate diffusion

In this section we use our tools to show that degenerate elliptic operators generate holomorphic semigroups on the real space $L^2(\Omega)$. We start by a 1–dimensional example.

Example 7.1. (degenerate diffusion in dimension 1.) Let $H = L^2(a, b)$, $-\infty \leq a < b \leq \infty$, be the real Hilbert space and let $\alpha, \beta, \gamma \in L_{\text{loc}}^\infty(a, b)$ be real coefficients. We assume that

$$\gamma^- \in L^\infty(a, b) \text{ and that } \beta^2(x) \leq c_1 \cdot \alpha(x) \quad (x \in (a, b))$$

for some constant $c_1 \geq 0$.

We define the bilinear form a on $L^2(a, b)$ by

$$a(u, v) = \int_a^b (\alpha(x)u'(x)v'(x) dx + \beta(x)u'(x)v(x) + \gamma(x)u(x)v(x)) dx$$

with domain

$$D(a) = H_c^2(a, b) .$$

Then the form a is *sectorial*, i.e. there exist constants $c \geq 0, \omega \in \mathbb{R}$ such that

$$|a(u, v) - a(v, u)| \leq c(a(u) + a(v)) + \omega(\|u\|_{L^2}^2 + \|v\|_{L^2}^2) .$$

As a consequence, letting A be the operator associated with a , we know that $-A$ generates a holomorphic C_0 -semigroup T on $L^2(\Omega)$. Moreover, T is submarkovian.

Proof. We use Young's inequality

$$|xy| \leq \varepsilon x^2 + \frac{1}{4\varepsilon} y^2$$

twice. On one hand we have for $\delta > 0$,

$$\begin{aligned} |a(u, v) - a(v, u)| &= \left| \int_a^b \beta(u'v - uv') dx \right| \\ &\leq \int_a^b (\delta\beta^2(u'^2 + v'^2) + \frac{1}{4\delta}(u^2 + v^2)) dx . \end{aligned}$$

Thus

$$\begin{aligned}
& c(a(u) + a(v)) + \omega(\|u\|_H^2 + \|v\|_H^2) = \\
& \int_a^b (c\alpha(u'^2 + v'^2) + c\beta(u'u + v'v) + (c\gamma + \omega)(u^2 + v^2)) \, dx \geq \\
& \int_a^b (c\alpha - \varepsilon\beta^2)(u'^2 + v'^2) - c^2 \frac{1}{4\varepsilon}(u^2 + v^2) + (c\gamma + \omega)(u^2 + v^2) \, dx \\
& \geq \int_a^b (c\alpha - \varepsilon\beta^2)(u'^2 + v'^2) + (\omega - c\|\gamma^-\|_{L^\infty} - \frac{c^2}{4\varepsilon})(u^2 + v^2) \, dx \\
& \geq |a(u, v) - a(v, u)|
\end{aligned}$$

if $(c\alpha - \varepsilon\beta^2) \geq \delta\beta^2$ and $(\omega - c\|\gamma^-\|_{L^\infty} - \frac{c^2}{4\varepsilon}) \geq \frac{1}{4\delta}$ for some $\delta > 0, \varepsilon > 0, \omega > 0$. This is certainly true if $\beta^2 \leq c_1\alpha$. \square

The condition $\beta^2 \leq c_1\alpha$ shows in particular, that $\{x \in (a, b) : \alpha(x) = 0\} \subset \{x \in (a, b) : \beta(x) = 0\}$. This one has to assume in any case since an operator of the form $\beta u'$ cannot generate a holomorphic semigroup.

A special case is the *Black-Scholes Equation*

$$u_t + \frac{\sigma^2}{2}x^2u_{xx} + rxu_x - ru = 0 .$$

We consider $H = L^2(0, \infty)$ and let $a(u, v) = \int_0^\infty (\frac{\sigma^2}{2}x^2u'v' + (\sigma - r)xu'v + ruv) \, dx$ with $D(a) = H_c^1(0, \infty)$.

It is not difficult to extend the example above to higher dimension:

Example 7.2. Let $\Omega \subset \mathbb{R}^d$ be open and let $a_{ij}, b_j, c \in L_{\text{loc}}^\infty(\Omega)$ be real coefficients, $i, j = 1, \dots, d$. Assume that $c^- \in L^\infty(\Omega)$, that $a_{ij} = a_{ji}$ and that

$$c_1A(x) - B^2(x) \text{ is positive semidefinite}$$

for some $c_1 > 0$ for almost all $x \in \Omega$, where

$$A(x) = (a_{ij}(x))_{i,j=1,\dots,d}, B(x) = \text{diag}(b_1(x), \dots, b_d(x)) .$$

Define the form a on $L^2(\Omega)$ by

$$a(u, v) = \int_\Omega \left\{ \sum_{i,j=1}^d a_{ij}(x) \partial_i u \partial_j v + \sum_{j=1}^d b_j \partial_j uv + cuv \right\} \, dx$$

with domain

$$D(a) = H_c^1(\Omega) .$$

Then a is sectorial. The associated semigroup T on $L^2(\Omega)$ is submarkovian.

This and the previous example incorporate Dirichlet boundary conditions. In the next one we consider a degenerate elliptic operator with Neumann boundary conditions.

Example 7.3. Let $\Omega \subset \mathbb{R}^d$ be an open, possibly unbounded subset of \mathbb{R}^d . Let $a_{ij} \in L^\infty(\Omega)$ be *real* coefficients such that for some $\theta \in [0, \pi/2)$,

$$\sum_{i,j=1}^d a_{ij} \xi_i \bar{\xi}_j \in \Sigma_\theta \quad (\xi \in \mathbb{C}^d) .$$

Consider the form a on $L^2(\Omega)$ given by

$$a(u, v) = \int \sum_{i,j=1}^d a_{ij} \partial_i u \partial_j v \, dx$$

with domain $D(a) = H^1(\Omega)$. Then a is sectorial. Let T be the associated semigroup. Our criteria show right away that T is submarkovian. It is remarkable that even

$$T_\infty(t)1_\Omega = 1_\Omega \quad (t \geq 0) .$$

For bounded Ω this is easy to prove, but otherwise more sophisticated tools are needed (see [AtE09]). Note that T extends consistently to semigroups T_p on $L^p(\Omega)$, $2 \leq p \leq \infty$, where T_p is strongly continuous for $p < \infty$ and T_∞ is the adjoint of a strongly continuous on $L^1(\Omega)$.

We want to add an abstract result which shows that our solutions are some kind of *viscosity solutions*. This is illustrated particularly well in the situation of Example 7.3.

Proposition 7.4. ([AtE09, Corollary 3.9]). *Let V, H be real Hilbert spaces such that $V \hookrightarrow H$. Let $a : V \times V \rightarrow \mathbb{R}$ be continuous and sectorial. Assume that $a(u) \geq 0$ for all $u \in V$. Let $b : V \times V \rightarrow \mathbb{R}$ be continuous and coercive. Thus for each $n \in \mathbb{N}$ the form*

$$a + \frac{1}{n}b : V \times V \rightarrow \mathbb{R}$$

is continuous and coercive. Let A_n be associated with $a + \frac{1}{n}b$ and A with a . Then

$$(A_n + \lambda)^{-1}f \rightarrow (A + \lambda)^{-1}f \text{ as } n \rightarrow \infty \text{ in } H$$

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for each $f \in H, \lambda > 0$. Moreover, denoting by T_n and T the semigroup generated by $-A_n$ and by $-A$ one has

$$\lim_{n \rightarrow \infty} T_n(t)f = T(t)f \text{ in } H$$

for all $f \in H$.

The point in the result is that the form a is merely sectorial and may be degenerate. For instance, in Example 7.3 $a_{ij}(x) = 0$ is allowed. If we perturb by the Laplacian, we obtain a coercive form

$$a_n : H^1(\Omega) \times H^1(\Omega) \rightarrow \mathbb{R}$$

given by

$$a_n(u, v) = a(u, v) + \frac{1}{n} \int_{\Omega} \nabla u \nabla v .$$

Then Proposition 7.4 says that in the situation of Example 7.3 for this perturbation one has $(A_n + \lambda)^{-1}f \rightarrow (A + \lambda)^{-1}f$ as $n \rightarrow \infty$ in $L^2(\Omega)$ for all $f \in L^2(\Omega)$.

8 The Dirichlet-to-Neumann operator

Let $\Omega \subset \mathbb{R}^d$ be a bounded open set with boundary $\partial\Omega$. Our point is that we do not need any regularity assumption on Ω . Still we are able to define the Dirichlet-to-Neumann operator on $L^2(\partial\Omega)$ and to show that it is selfadjoint and generates a submarkovian semigroup on $L^2(\Omega)$. Formally, the Dirichlet-to-Neumann operator D_0 is defined as follows. Given $\varphi \in L^2(\Gamma)$, one solves the Dirichlet problem

$$\begin{cases} \Delta u &= 0 \text{ in } \Omega \\ u|_{\partial\Omega} &= \varphi \end{cases}$$

and defines $D_0\varphi = \frac{\partial u}{\partial\nu}$. We will give a precise definition using weak derivatives. We consider the space $L^2(\partial\Omega) := L^2(\partial\Omega, \mathcal{H}^{d-1})$ with the $(d-1)$ -dimensional Hausdorff measure \mathcal{H}^{d-1} . Integrals over $\partial\Omega$ are always taken with respect to \mathcal{H}^{d-1} , those over Ω always with respect to the Lebesgue measure.

Definition 8.1. (normal derivative). Let $u \in H^1(\Omega)$ such that $\Delta u \in L^2(\Omega)$. We say that

$$\frac{\partial u}{\partial\nu} \in L^2(\partial\Omega)$$

if there exists $g \in L^2(\partial\Omega)$ such that

$$\int_{\Omega} \Delta u v + \int_{\Omega} \nabla u \nabla v = \int_{\partial\Omega} g v$$

for all $v \in H^1(\Omega)$. This determines g uniquely and we let $\frac{\partial u}{\partial\nu} := g$.

Recall, that for $u \in L^1_{\text{loc}}(\Omega)$ the Laplacian Δu is defined in the sense of distributions. If $\Delta u = 0$, then $u \in C^\infty(\Omega)$ by elliptic regularity. Next we define traces of a function $u \in H^1(\Omega)$.

Definition 8.2. (traces). Let $u \in H^1(\Omega)$. We let

$$\begin{aligned} tr(u) = & \{g \in L^2(\Omega) : \exists u_n \in H^1(\Omega) \cap C(\bar{\Omega}) \\ & \lim_{n \rightarrow \infty} u_n = u \text{ in } H^1(\Omega) \text{ and} \\ & \lim_{n \rightarrow \infty} u_n|_{\partial\Omega} = g \text{ in } L^2(\partial\Omega)\} . \end{aligned}$$

For arbitrary open sets the trace is not unique. However, if Ω is a Lipschitz domain, then each $u \in H^1(\Omega)$ has a unique trace $u|_{\partial\Omega} \in L^2(\partial\Omega)$. Now we are in the position to define the Dirichlet-to-Neumann operator D_0 . Its domain is given by

$$\begin{aligned} D(D_0) := & \{\varphi \in L^2(\partial\Omega) : \exists u \in H^1(\Omega) \text{ such that } \Delta u = 0 , \\ & \frac{\partial u}{\partial \nu} \in L^2(\partial\Omega) \text{ and} \\ & \text{such that } \varphi \in tr(u)\} \end{aligned}$$

and we define

$$D_0 = \frac{\partial u}{\partial \nu}$$

where $u \in H^1(\Omega)$ is such that $\Delta u = 0$, $\frac{\partial u}{\partial \nu} \in L^2(\partial\Omega)$, $\varphi \in tr(u)$. It is part of our result that this operator is univocal.

Theorem 8.3. *The operator D_0 is selfadjoint and $-D_0$ generates a submarkovian semigroup on $L^2(\partial\Omega)$.*

In the proof we use Theorem 6.4. Here a non-injective mapping j is needed. We also need Maz'ya's inequality. Let $q = \frac{2d}{d-1}$. There exists a constant c_M such that

$$\left(\int_{\Omega} |u|^q \right)^{2/q} \leq c_M \left(\int_{\Omega} |\nabla u|^2 + \int_{\partial\Omega} |u|^2 \right)$$

for all $u \in C(\bar{\Omega}) \cap H^1(\Omega)$.

Proof of Theorem 8.3. We consider real spaces. Our Hilbert space is $L^2(\partial\Omega)$. Let $D(a) = C(\bar{\Omega}) \cap H^1(\Omega)$, $a(u, v) = \int_{\Omega} \nabla u \nabla v$, $j(u) = u|_{\partial\Omega} \in L^2(\partial\Omega)$. Then a is symmetric and $a(u) \geq 0$ for all $u \in D(a)$. Thus the sectoriality condition before Theorem 6.4 is trivially satisfied. Denote by A the operator on $L^2(\partial\Omega)$ associated with (a, j) . Let $\varphi, \psi \in L^2(\partial\Omega)$. Then $\varphi \in D(A)$ and $A\varphi = \psi$ if and only if there exist $u_n \in C(\bar{\Omega}) \cap H^1(\Omega)$ such that $u_n|_{\partial\Omega} \rightarrow \varphi$ in $L^2(\partial\Omega)$, $a(u_n, v) \rightarrow \int_{\partial\Omega} \psi v|_{\partial\Omega}$ and $\lim_{n, m \rightarrow \infty} \int_{\Omega} |\nabla(u_n - u_m)|^2 = 0$. Now Maz'ya's inequality implies that $(u_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $H^1(\Omega)$. Thus $\lim_{n \rightarrow \infty} u_n =$

u exists in $H^1(\Omega)$, and so $\varphi \in \text{tr}(u)$. Moreover $\int_{\partial\Omega} \psi v = \lim_{n \rightarrow \infty} \int_{\Omega} \nabla u_n \nabla v = \int_{\Omega} \nabla u \nabla v$ for all $v \in H^1(\Omega)$. Taking as v test functions, we see that $\Delta u = 0$. Thus

$$\int_{\Omega} \nabla u \nabla v + \int_{\Omega} \Delta u v = \int_{\partial\Omega} \psi v$$

for all $v \in H^1(\Omega)$. Consequently, $\frac{\partial u}{\partial \nu} = \psi$. We have shown that $A \subset D_0$. Conversely, let $\varphi \in D(D_0)$, $D_0 \varphi = \psi$. Then there exists $u \in H^1(\Omega)$ such that $\Delta u = 0$, $\varphi \in \text{tr}(u)$ and $\frac{\partial u}{\partial \nu} = \psi$. Since $\varphi \in \text{tr}(u)$ there exist $u_n \in C(\bar{\Omega}) \cap H^1(\Omega)$ such that $u_n \rightarrow u$ in $H^1(\Omega)$ and $u_n|_{\partial\Omega} \rightarrow \varphi$ in $L^2(\partial\Omega)$. It follows that $j(u_n) = u_n|_{\Gamma} \rightarrow \varphi$ in $L^2(\partial\Omega)$, $(a(u_n))_{n \in \mathbb{N}}$ is bounded and

$$a(u_n, v) = \int_{\Omega} \nabla u_n \nabla v \rightarrow \int_{\Omega} \nabla u \nabla v = \int_{\Omega} \nabla u \nabla v + \int_{\Omega} \Delta u v = \int_{\partial\Omega} \psi v$$

for all $v \in H^1(\Omega) \cap C(\bar{\Omega})$. Thus, $\varphi \in D(A)$ and $A\varphi = \psi$ by the definition of the associated operator. Since a is symmetric A is selfadjoint. Now the claim follows from Theorem 6.4.

Our criteria easily apply and show that semigroup generated by $-D_0$ is submarkovian. \square

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